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On generalized variational inequalities

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Abstract We obtain a new version of the minimax inequality of Ky Fan. As an application, an existence result for the generalized variational inequality problem with set-valued mappings defined on noncompact sets in Hausdorff topological vector spaces is given. Also, some existence results for the generalized variational inequality problem for quasimonotone and pseudomonotone mappings are obtained.

Keywords Fan's KKM theorem · Minimax inequality · Variational inequality

Mathematics Subject Classification (2000) 47H04 · 47H05 · 49J40 · 49J53

1 Introduction

There have been significant developments in the theory of optimization techniques in the recent decades. The study of variational inequalities is also a part of this development, because optimization problems can often be reduced to the solution of variational inequalities. Other applications of the variational inequality problem are in nonlinear elliptic boundary value problems, mathematical programming, mathematical economics and many other areas. Several authors [3–7,16,17,21–23] have proved many interesting results on the variational

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inequality problem. The most general and popular forms of the variational inequality with very reasonable conditions are due to Browder [6] and Kinderleher and Stampacchia [22].

Our purpose in this paper is to study variational inequalities and generalized variational inequalities by a direct use of the generalized minimax inequality of Fan. Let X be a topological vector space (t.v.s.) with dual X^* and K be a nonempty convex subset of X. Suppose that $T : K \rightrightarrows X^*$ is a set-valued mapping and $F : K \to X^*$ is a single-valued mapping.

The variational inequality (in short, VI(K, F)) is: find $\bar{y} \in K$ such that

$$\langle F(\bar{y}), \bar{y} - x\rangle \leq 0, \quad \forall x \in K.$$

The generalized variational inequality (in short, GVI(K, T)) is: find $\bar{y} \in K$ and $\bar{y^*} \in T(\bar{y})$ such that

$$\langle \bar{y^*}, \bar{y} - x\rangle \leq 0, \quad \forall x \in K.$$

We now recall some standard definitions and notations which will be used in the sequel.

If E is a nonempty subset of X, then we denote by $\mathcal{F}(E)$ the family of all nonempty finite subsets of E. Let K_0 be a nonempty subset of K. A set-valued map $\Gamma : K_0 \rightrightarrows K$ is called a KKM map if for each $A \in \mathcal{F}(K_0)$, $\text{conv}(A) \subseteq \bigcup_{x\in A} \Gamma(x)$. Let Y be a nonempty set. Then, $\Gamma : Y \rightrightarrows K$ is said to be transfer closed-valued if for any $(y, x) \in Y \times K$ with $x \notin \Gamma(y)$ there exists $y' \in Y$ such that $x \notin \text{cl}_K \Gamma(y')$. If $Y = K$, then we will call Γ transfer closed-valued on K. If $A \subseteq K$, then a map $\Gamma : K \rightrightarrows K$ is called transfer closed-valued on A if the map $y \mapsto \Gamma(y) \bigcap A$, $y \in A$, is transfer closed-valued.

Suppose that f is a real-valued bifunction on $Y \times K$. Then, we say that f is transfer lower semicontinuous(l.s.c.) in the second variable if for each $(y, x) \in Y \times K$ with $f(y, x) > 0$ there exists $y' \in Y$ and a neighborhood $U(x)$ of x in K such that $f(y', z) > 0$ for all $z \in U(x)$. If $Y = K$ and $A \subseteq K$, then we call f transfer l.s.c. in the second variable on A, if $f\mid_{A\times A}$ is transfer l.s.c. in the second variable. Let $f : K \times K \to \mathbb{R}$. We recall that:

(i) f is properly quasimonotone [2] if, $\min_i f(x_i, \overline{x}) \leq 0$, for all $\{x_1, ..., x_n\} \subset K$, $\bar{x} \in \text{conv}(\{x_1, ..., x_n\})$.
(ii) f is called 0-segmentary closed if, $\forall x, y \in K$, when (y_α) be a net on K converging to y, then the following implication holds: if $f(u, y_\alpha) \leq 0$ for all $u \in [x, y]$, then $f(x, y) \leq 0$.

Suppose that $T : K \rightrightarrows X^*$ is a set-valued mapping, then T is said to be

(i) upper sign-continuous [16] if, for all $x, y \in K$, the following implication holds:

$$\inf_{x^*\in T(u)} \langle x^*, y - x\rangle \geq 0, \quad \forall u \in\ ]x, y[\Rightarrow \quad \sup_{x^*\in T(x)} \langle x^*, y - x\rangle \geq 0;$$

(ii) pseudomonotone [20] if, for every $x, y \in K$ and every $x^* \in T(x)$, $y^* \in T(y)$, the following implication holds:

$$\langle x^*, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0;$$

(iii) quasimonotone [18] if, for every $x, y \in K$ and every $x^* \in T(x)$, $y^* \in T(y)$, the following implication holds:

$$\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq 0;$$

(iv) properly quasimonotone [10] if, for all $x_1, ..., x_n$ and all $x \in \text{conv}\{x_1, ..., x_n\}$, there exists $i \in \{1, 2, ..., n\}$ such that:

$$\forall x^* \in T(x_i) : \langle x^*, x_i - x\rangle \geq 0.$$

(v) 0-segmentary closed if the function f defined as follows:

$$f(x, y) = \inf_{y^* \in T(y)} \langle y^*, y - x\rangle$$

is 0-segmentary closed.

Notice that every pseudomonotone set-valued mapping with nonempty, weak*-compact values is properly quasimonotone and also a properly quasimonotone operator is quasimonotone [10].

Brézis, et al. [5] improved Fan's KKM lemma [14] by assuming the closedness condition only on finite dimensional subspaces, with some topological pseudomonotone condition. Chowdhury and Tan [8], replacing finite dimensional subspaces by polytopes, restated the Brézis, Nirenberg and Stampacchia result under weaker assumptions. Ding and Tarafdar [11] obtained the result of Chowdhury and Tan under weaker compactness condition. The Chowdhury and Tan's result was also proved by Kalmoun [19] for transfer closed-valued multi-valued mappings. In [12,13], the authors refined the Ding and Tarafdar's result and the Kalmoun's result. Based on remark 2 in [5], we give a short and direct proof of the following version of the above results in the Appendix.

Lemma 1.1 *Let K be a nonempty and convex subset of a Hausdorff t.v.s. X. Suppose that $\Gamma : K \rightrightarrows K$ is a set-valued mapping such that the following conditions are satisfied:*

(A1) *Γ is a KKM map,*
(A2) *for all $A \in \mathcal{F}(K)$, Γ is transfer closed-valued on conv(A),*
(A3) *for all $x, y \in K$, $cl_K(\bigcap_{u\in[x,y]} \Gamma(u)) \cap [x, y] = (\bigcap_{u\in[x,y]} \Gamma(u)) \cap [x, y]$,*
(A4) *there is a nonempty compact convex set $B \subseteq K$, such that $cl_K(\bigcap_{x\in B} \Gamma(x))$ is compact.*
Then, $\bigcap_{x\in K} \Gamma(x) \neq \emptyset$.

We shall need the following Lemma.

Lemma 1.2 [11] *Let X be a topological vector space and A be a finite subset of X. Assume that $T : conv(A) \rightrightarrows X^*$ is upper semicontinuous with weak*-compact values. Let $\phi : conv(A) \times conv(A) \to \mathbb{R}$ be defined by*

$$\phi(x, y) = \inf_{y^* \in T(y)} \langle y^*, y - x\rangle.$$

Then for any fixed $x \in conv(A)$, the function $y \mapsto \phi(x, y)$ is lower semicontinuous on $conv(A)$.

2 Main results

In this section, we first apply Lemma 1.1 to obtain a generalized minimax inequality of Ky Fan. Then by using this result we obtain some existence results for the variational inequality problem for pseudomonotone mappings and the generalized variational inequality problem with set-valued mappings defined on noncompact sets in Hausdorff topological vector spaces. In the first step we obtain the following result which improves Theorem 1 of [4].

Theorem 2.1 *Suppose that K is a nonempty and convex subset of a Hausdorff t.v.s. X. Let B be a nonempty, convex and compact subset of K with $core_K(B) \neq \emptyset$ and $f : K \times K \to \mathbb{R}$ be a real-valued bifunction such that:*

(i) *f is properly quasimonotone on B and $f(x,x) = 0$ for any $x \in B$;*
(ii) *f is transfer l.s.c. in the second argument on conv(A) for any $A \in \mathcal{F}(B)$;*
(iii) *f is 0-segmentary closed;*
(iv) *if $f(x,y) > 0$ and $f(z,y) = 0$, then $f(u,y) > 0$, for all $u \in]x,z[$;*
(v) *for all $y \in B \backslash core_K(B)$ there exists $x \in core_K(B)$ such that $f(x,y) \geq 0$.*

Then, there exists $\bar{y} \in B$ such that $f(x,\bar{y}) \leq 0$ for all $x \in K$.

Proof Suppose that $\Gamma : B \rightrightarrows B$ is defined as follows:

$$\Gamma(x) = \{y \in B : f(x,y) \leq 0\}.$$

It is easily seen that all the conditions of Lemma 1.1 are fulfilled. Indeed, condition (i) implies that Γ is a KKM map. From condition (ii) we obtain Γ is transfer closed-valued on conv(A). Condition (iii) implies condition (A3). Therefore, there exists $\bar{y} \in B$ such that

$$f(x,y) \leq 0 \quad \forall x \in B. \tag{1}$$

Now, we show that $f(x,\bar{y}) \leq 0$ for all $x \in K$. Suppose in the contrary there exists $x_0 \in K$ such that $f(x_0,\bar{y}) > 0$. If $\bar{y} \in core_K(B)$, then there exists $0 < t < 1$ such that $x_t = t\bar{y} + (1-t)x_0 \in B$. Since $f(\bar{y},\bar{y}) = 0$ and $f(x_0,\bar{y}) > 0$, then by condition (iv) we have $f(x_t,\bar{y}) > 0$ which contradicts (1). If $\bar{y} \notin core_K(B)$, then by condition (v) there exists $\bar{x} \in core_K(B)$ such that $f(\bar{x},\bar{y}) \geq 0$. Hence, from (1) we have $f(\bar{x},\bar{y}) = 0$. Moreover, there exists $t_0 \in (0,1)$ such that $x_{t_0} = t_0\bar{x} + (1-t_0)x_0 \in B$. But from condition (iv) we have $f(x_{t_0},\bar{y}) > 0$ which contradicts (1).

From the above theorem, we obtain an existence result for set-valued mappings defined on noncompact sets.

Theorem 2.2 *Let K be a nonempty and convex subset of a Hausdorff t.v.s. X. Let B be a nonempty, convex and compact subset of K with $core_K(B) \neq \emptyset$. Suppose that $T : K \rightrightarrows X^*$ is a set-valued mapping with nonempty weak*-compact values such that:*

(a) *for each $A \in \mathcal{F}(B)$, T is u.s.c. from conv(A) to X^* provided with the weak*- topology;*
(b) *T is 0-segmentary closed;*
(c) *for all $y \in B \backslash core_K(B)$ there exists $x \in core_K(B)$ such that* $\inf_{y^* \in T(y)} \langle y^*, y - x\rangle \geq 0$.

Then there exists a point $\bar{y} \in K$ such that

$$\inf_{y^* \in T(\bar{y})} \langle y^*, \bar{y} - x\rangle \leq 0 \quad \forall x \in K.$$

If in addition, $T(\bar{y})$ is convex, then the problem GVI(K,T) has a solution.

Proof Suppose that $f : K \times K \to \mathbb{R}$ is defined as follows:

$$f(x,y) := \inf_{y^* \in T(y)} \langle y^*, y - x\rangle \quad \forall x,y \in K.$$

Then by Lemma 1.2, f is lower semicontinuous in the second argument on conv(A) for any $A \in \mathcal{F}(B)$. Also, we can easily show that f satisfies conditions (i) and (iv) of Theorem 2.1 Conditions (b) and (c) imply conditions (iii) and (v) of Theorem 2.1, respectively. Hence, there exists $\bar{y} \in B$ such that

$$\inf_{y^* \in T(\bar{y})} \langle y^*, \bar{y} - x\rangle \leq 0 \quad \forall x \in K.$$

Now suppose that $T(\bar{y})$ is convex. If $g : K \times T(\bar{y}) \to \mathbb{R}$ is defined as follows:

$$g(x, y^*) = \langle y^*, \bar{y} - x \rangle,$$

then by Kneser's minimax theorem we have

$$\inf_{y^* \in T(\bar{y})} \sup_{x \in K} \langle y^*, \bar{y} - x \rangle = \sup_{x \in K} \inf_{y^* \in T(\bar{y})} \langle y^*, \bar{y} - x \rangle \leq 0.$$

Therefore, there exists a point $\bar{y}^* \in T(\bar{y})$ such that

$$\sup_{x \in K} \langle \bar{y}^*, \bar{y} - x \rangle \leq 0.$$

As a consequence of the above theorem, we obtain the following corollary which improves Theorems 2.1 and 2.3 of [21] without degree theory. Moreover, if $T : K \rightrightarrows X^*$ has an upper-sign continuous selection, then we deduce Theorem 2.1 of [21]. In fact in Theorem 2.1 in [21], $K \subseteq \mathbb{R}^n$ and T has a continuous selection.

Corollary 2.1 *Let K be a nonempty, convex, bounded and closed subset of a reflexive Banach space X. Suppose that $T : K \rightrightarrows X^*$ is u.s.c. with nonempty weak*-compact convex values. Consider the following statements:*

(*I*) *there exists a reference vector $x^{ref} \in K$ such that the set*

$$L_<(T, x^{ref}) := \{x \in K : \inf_{x^* \in T(x)} \langle x^*, x - x^{ref} \rangle < 0\}$$

is bounded (possibly empty);

(*II*) *there exist an open ball Ω and a vector $x^{ref} \in \Omega \cap K$ such that*

$$\inf_{x^* \in T(x)} \langle x^*, x - x^{ref} \rangle \geq 0, \quad \forall x \in K \cap \partial\Omega,$$

where $\partial\Omega$ denotes the boundary of Ω;

(*III*) *the problem GVI(K, T) has a solution.*

Then $(I) \Rightarrow (II) \Rightarrow (III)$.

Proof If (I) holds, then there exists an open ball denoted by Ω such that

$$L_<(T, x^{ref}) \cup \{x^{ref}\} \subseteq \Omega \quad \text{and} \quad \partial\Omega \cap L_<(T, x^{ref}) = \emptyset.$$

Then we obtain the result. If (II) holds, then it is enough in Theorem 2.2 to set $B = cl\Omega \cap K$.

In the sequel we establish an existence result for quasimonotone operators.

We need the following version of Theorem 2.1 in [1] which is proved in [[15], Theorem 2.13].

Theorem 2.3 [15] *Let K be a nonempty convex subset of Hausdorff t.v.s. X and $T : K \rightrightarrows X^*$ be quasimonotone and upper sign-continuous with nonempty, weak*-compact values. Assume that there exist a nonempty compact subset $D \subseteq K$ and a nonempty, convex, compact $B \subseteq K$ such that for each $y \in K \backslash D$ there is $x \in B$ such that $\sup_{x^* \in T(x)} \langle x^*, y - x \rangle < 0$. Then, GVI$(K, T)$ has a solution.*

Theorem 2.4 *Suppose that K is a nonempty convex subset of a t.v.s. X and $T : K \rightrightarrows X^*$ is a quasimonotone operator which is upper sign-continuous with nonempty, convex, weak*-compact values. Assume that there exists a nonempty, compact and convex subset B of K with $core_K(B) \neq \emptyset$ such that for any $y \in B \backslash core_K(B)$, there exists $x \in core_K(B)$ such that $\inf_{y^* \in T(y)} \langle y^*, y - x\rangle \geq 0$. Then GVI$(K, T)$ has a solution.*

Proof By Theorem 2.3, GVI(B, T) has a solution. Suppose that $\bar{y} \in B$ and $\bar{y}^* \in T(\bar{y})$ such that

$$\langle \bar{y}^*, \bar{y} - x\rangle \leq 0 \quad \forall x \in B. \tag{2}$$

We will show that in fact $\bar{y}$ is a solution for GVI(K, T). If $\bar{y} \in core_K(B)$, then for any $x \in K \backslash B$, there exists $t \in (0, 1)$ such that $(1-t)x + t\bar{y} \in B$, thus from 2, we deduce that

$$\langle \bar{y}^*, (1-t)x + t\bar{y} - \bar{y}\rangle = \langle \bar{y}^*, (1-t)(x - \bar{y})\rangle \geq 0.$$

Hence, we obtain

$$\langle \bar{y}^*, x - \bar{y}\rangle \geq 0, \quad \forall x \in K.$$

If $\bar{y} \in B \backslash core_K(B)$, then from assumption, there exists $\bar{x} \in core_K(B)$ such that

$$\langle \bar{y}^*, \bar{y} - \bar{x}\rangle \geq 0,$$

but by (2), we have

$$\langle \bar{y}^*, \bar{y} - \bar{x}\rangle \leq 0.$$

Hence,

$$\langle \bar{y}^*, \bar{y}\rangle = \langle \bar{y}^*, \bar{x}\rangle. \tag{3}$$

If $x \in K \backslash B$, then there exists $t \in (0, 1)$ such that $t\bar{x} + (1-t)x \in B$. Therefore, from (2) and (3), we obtain

$$\langle \bar{y}^*, (1-t)(x - \bar{y})\rangle = \langle \bar{y}^*, t\bar{x} + (1-t)x - \bar{y}\rangle \geq 0.$$

Hence,

$$\langle \bar{y}^*, \bar{y} - x\rangle \leq 0, \quad \forall x \in K.$$

Remark 2.1 From the above result, we can deduce the main result of [[17], Theorem 3.1] for both bounded and unbounded K.

As a consequence of the above theorem we have the following result which improves Theorem 3.1 of [21]. In fact in Theorem 3.1 of [21], F is continuous on finite dimensional subspaces of X.

Corollary 2.2 *Let K be a closed convex subset of a reflexive Banach space X. Assume that $F : K \to X^*$ is pseudomonotone and upper sign-continuous. Then the following statements are equivalent:*

(a) *There exists a reference vector $x^{ref} \in K$ such that the set*

$$L_<(F, x^{ref}) := \{x \in K : \langle F(x), x - x^{ref}\rangle < 0\}$$

is bounded (possibly empty)

(b) *There exist an open ball* Ω *and a vector* $x^{ref} \in \Omega \cap K$ *such that*

$$\langle F(x), x - x^{ref} \rangle \geq 0, \quad \forall x \in K \cap \partial\Omega.$$

(c) *The problem VI*(K, F) *has a solution.*

Proof For (a)$\Rightarrow$(b) we do the same as in the proof of Corollary 2.1. If (b) holds, then it is enough in Theorem 2.4 to set $B = cl\Omega \cap K$.

For (c)$\Rightarrow$(a), let x^{ref} be a solution of VI(K, F). Then

$$\langle F(x^{ref}), x - x^{ref} \rangle \geq 0, \quad \forall x \in K.$$

Hence, pseudomonotonicity of F implies that

$$\langle F(x), x - x^{ref} \rangle \geq 0, \quad \forall x \in K.$$

Therefore, $L_<(F, x^{ref}) = \emptyset$ and (a) holds.

Remark 2.2

(1) The equivalence between (a) and (c) of Corollary 2.2 is proved in Theorem 3.2 of [3]. Furthermore, the case of single-valued quasimonotone maps is also studied.
(2) It is possible to generalize the above corollary for a set-valued mapping T. In fact we can only assume that T has a continuous selection. Therefore, we can obtain Theorems 2.2 and 3.2 of [21]. In fact in these results T has an upper sign-continuous selection. As it is shown by examples in [3] and [21] there is no similar result for quasimonotone cases.

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Appendix

We give a short and direct proof of Lemma 1.1.

Proof We will establish the result in two steps. First step: Let K be compact. Suppose that $A \in \mathcal{F}(K)$ and $\Gamma_A : \text{conv}(A) \rightrightarrows \text{conv}(A)$ be defined as $\Gamma_A(x) := \text{cl}_{\text{conv}(A)}(\Gamma(x) \cap \text{conv}(A).\Gamma_A$ is a KKM mapping and it is nonempty and compact-valued. It follows from Fan's KKM theorem that

$$\bigcap_{x \in \text{conv}(A)} \Gamma_A(x) \neq \emptyset.$$

Hence by (A2), we obtain

$$\bigcap_{x \in \text{conv}(A)} \Gamma(x) \cap \text{conv}(A) \neq \emptyset. \tag{4}$$

Let $\{D_i\}_{i \in I}$ be the family of all convex hulls of finite subsets of K partially ordered by $\subseteq$. By (4), $M_{D_i} = \bigcap_{x \in D_i} \Gamma(x) \cap D_i$ for each $i \in I$ is nonempty. Take any $u_i \in M_{D_i}$ for each $i \in I$ and let $E_i = \{u_j : j \geq i\}$. Clearly $\{E_i : i \in I\}$ has the finite intersection property.

Hence, $\cap_{i\in I}\mathrm{cl}_K E_i \neq \emptyset$. Take any $\bar{x} \in \cap_{i\in I}\mathrm{cl}_K E_i$. Note that for any $i \in I$ and any $j \in I$ with $j \geq i$,

$$u_j \in M_{D_j} \subseteq \bigcap_{x\in D_i} \Gamma(x) \cap D_j \subseteq \bigcap_{x\in D_i} \Gamma(x).$$

Therefore, $E_i \subseteq \bigcap_{x\in D_i} \Gamma(x)$. If $x \in K$, there exists $i_0 \in I$ such that $\bar{x}, x \in D_{i_0}$ and therefore $[\bar{x}, x] \subseteq D_{i_0}$. Hence,

$$\bar{x} \in \mathrm{cl}_K E_{i_0} \subseteq \mathrm{cl}_K \left(\bigcap_{u\in D_{i_0}} \Gamma(u)\right) \subseteq \mathrm{cl}_K \left(\bigcap_{u\in[\bar{x},x]} \Gamma(u)\right).$$

Thus, by condition (A3)

$$\bar{x} \in \mathrm{cl}_K \left(\bigcap_{u\in[\bar{x},x]} \Gamma(u)\right) \cap [\bar{x}, x] = \left(\bigcap_{u\in[\bar{x},x]} \Gamma(u)\right) \cap [\bar{x}, x].$$

Therefore, $\bar{x} \in \Gamma(x)$ for all $x \in K$ and hence

$$\bigcap_{x\in K} \Gamma(x) \neq \emptyset.$$

Second step: Let K be an arbitrary nonempty convex subset of X. Suppose that $A \in \mathcal{F}(K)$, and $L_A = \mathrm{conv}(A \bigcup B)$, then L_A is compact. Let $\Gamma_A : L_A \rightrightarrows L_A$ be defined as $\Gamma_A(x) = \Gamma(x) \cap L_A$. Then by the above argument We obtain

$$\bigcap_{x\in L_A} \Gamma_A(x) \neq \emptyset.$$

Now, assume that

$$M_A = \bigcap_{x\in L_A} \Gamma(x) \text{ forany } A \in \mathcal{F}(K), \tag{5}$$

then

$$M_A \subseteq \bigcap_{x\in B} \Gamma(x) \text{ forall } A \in \mathcal{F}(K). \tag{6}$$

If $\mathcal{M} = \{M_A : A \in \mathcal{F}(K)\}$, then by (5) one can see that the class $\mathcal{M}$ has the finite intersection property. Therefore, from (6) and (A4), we have

$$\bigcap_{A\in\mathcal{F}(K)} \mathrm{cl}_K M_A \neq \emptyset.$$

If $\bar{x} \in \bigcap_{A\in\mathcal{F}(K)} \mathrm{cl}_K M_A$, $x \in X$ and $A_0 = \{\bar{x}, x\}$, then $\mathrm{conv} A_0 = [\bar{x}, x]$ and

$$\bar{x} \in \mathrm{cl}_K M_{A_0} = \mathrm{cl}_K \left(\bigcap_{u\in L_{A_0}} \Gamma(u)\right) \subseteq \mathrm{cl}_K \left(\bigcap_{u\in[\bar{x},x]} \Gamma(u)\right)$$

Hence, by condition (A3)

$$\bar{x} \in \mathrm{cl}_K \left(\bigcap_{u \in [\bar{x}, x]} \Gamma(u) \right) \cap [\bar{x}, x] = \left(\bigcap_{u \in [\bar{x}, x]} \Gamma(u) \right) \cap [\bar{x}, x].$$

Therefore, $\bar{x} \in \Gamma(x)$ for all $x \in X$ and the proof is complete.

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